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Calculation of the Self-Consistent Electric Field in Toroidal Nonaxisymmetric Devices*

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In a toroidal nonaxisymmetric plasma, the radial electric field is determined by the constraint that the radial ion and electron fluxes be equal. This ambipolarity relationship is in general a nonlinear algebraic equation for the electric field that can have multiple solutions. An algorithm is proposed here to solve this equation and obtain a spatially continuous, temporally stable solution. For definiteness, this is applied to a bumpy torus, and it is shown that there exists a boundary in the density, electron temperature, and ion temperature space across which the potential changes abruptly from a spatial hill to a spatial well.

I. INTRODUCTION

For a toroidal nonaxisymmetric device it is supposed [1, 2] that the radial electric field can be determined from the constraint of steady-state ambipolarity. That is, this model posits that at every spatial location, the plasma reacts to ensure that the ion flux balances the electron flux. This determines the electric field, since if the fluxes are written as functions of the field, then ion and electron flux equality can only be satisfied at a finite number of values of the field. This is in contrast to the case for an axisymmetric tokamak, where the fluxes are equal for all values of the electric field.

In general, this ambipolarity relationship is a nonlinear algebraic equation for the electric field (except possibly if the fluxes are deep in the large-orbit banana regime, where the orbit width depends on the spatial derivative of the electric field and, hence, the ambipolarity relationship is differential). This nonlinear algebraic

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equation can have multiple solutions for some ranges of the density and temperature space. It can be shown for a given density (and density gradient) that if multiple solutions exist, then at least one of them is an unstable solution, while the others are stable. By this we mean that a small perturbation away from the unstable solution will grow while a small perturbation away from the attractive solutions will decay.

For the multiple stable steady-state solutions the conceptual problem arises as how the plasma chooses which solution to exist on. In addition, the problem is compounded by the fact that the algebraic equation is spatially localized, so there is no explicit spatial connection between points. This implies that a spatially discontinuous electric field could be the solution of the steady-state ambipolarity relationship.

There have been several attempts to resolve these conceptual difficulties. The earliest [1] was to solve the time-dependent ambipolarity relationship. This procedure automatically chooses the stable solutions and indicates that the electric field solution found at any given radial location will be the stable solution closest to the initial data. However, this still has the problem that if the initial data are such that at one point a stable solution is found and at another point a different stable solution is found, then the electric field profile will be discontinuous. Recently [3, 4], a second-order diffusion equation for the electric field was derived by extending the fluxes through fourth order in the deviation from a flux surface. This explicitly imposes continuity on the electric field solution and corresponds to the physically appealing notion that neighboring flux surfaces are linked through the finite orbit deviation from the flux surface. However, this approach has a number of problems. The first is that the fourth-order contribution to the flux is derived perturbatively, but when it is important, then the perturbation theory has broken down. The second problem is that constructing a second-order differential equation requires two boundary conditions for a complete solution. The boundary condition at the center of the plasma is obvious; that is, the electric field must be zero there from considerations of symmetry. The boundary condition at the edge of the plasma is not at all obvious and there is no clearly thought out reason to choose a given condition. Not surprisingly, detailed numerical experimentation with the diffusion equation has shown that the choice of the outer boundary condition can have a marked impact on the overall shape of the potential.

In this paper, we outline a method for solving the steady-state ambipolarity equation in conjunction with the density and temperature equations. This done so as to impose the physical requirement of spatial continuity of the electric field arising from the finite orbit width.

In Section II, we describe our basic method; then in Section III we apply it for definiteness to a bumpy torus plasma and show that we can obtain continuous solutions. We also obtain a boundary in the electron-temperature, ion-temperature space across which the character of the potential changes. Finally, in Section IV, we discuss what future areas of research might be.

II. The Basic Method for Solving the Ambipolarity Relationship

The time-dependent ambipolarity relationship is given by [1]

$$\frac{\varepsilon_\perp}{e}\frac{\partial}{\partial t}\Phi' = \sum_a Z_a \Gamma_a(\Phi'; n_a, n_a', T_a, T_a'), \tag{1}$$

where the prime (′) means $\partial/\partial r$ and r is the radial flux variable defined by $\frac{1}{2}r^2 B_0 = \alpha$, with B_0 the magnetic field on the magnetic axis and α the toroidal magnetic flux that labels the pressure surfaces. The variable Φ is the electrostatic potential (hence the radial electric field is $E_r = -\partial\Phi/\partial r$). The dielectric function $\varepsilon_\perp$ arises mainly from the polarization drift of the particles in the time-varying electric field. In a bumpy torus $\varepsilon_\perp \simeq c^2\varepsilon_0 4\pi n_i m_i/B_0^2$, where m_i is the ion mass, c is the speed of light, and ε_0 is the free space dielectric constant. The charge number of species a is Z_a; n_a is the density and T_a the temperature of species a. The flux of species a is

$$\Gamma_a = -D_{n_a}(\Phi')\frac{\partial n_a}{\partial r} - D_{t_a}(\Phi')\frac{\partial T_a}{\partial r} - Z_a D_{n_a}(\Phi')\frac{n_a}{T_a}\Phi'. \tag{2}$$

We can nondimensionalize Eq. (1) by defining $\tau_E = \tau_p[\varepsilon_\perp T/(ena_p^2)]$, $\tau_p = a_p^2/D_{n0}$, $\bar{E} = -a_p\Phi'/T$, where a_p is the plasma radius and n, T, and D_{n0} are a reference density, temperature, and diffusivity such that $\Gamma_a/(D_{n0}n/a_p) = O(1)$. We have

$$\frac{\partial \bar{E}}{\partial t} = -\frac{1}{\tau_E}\left(\frac{a_p}{D_{n0}n}\right)\sum_a Z_a \Gamma_a(\bar{E}). \tag{3}$$

If we linearize Eq. (3) about its steady-state solution by writing $\bar{E} = E_0 + \tilde{E}$, where $\tilde{E} \ll E_0$, we have

$$\frac{\partial \tilde{E}}{\partial t} = -\left[\frac{1}{\tau_E}\left(\frac{a_p}{D_{n0}n}\right)\sum_a Z_a \frac{\partial \Gamma_a}{\partial \bar{E}}(E_0)\right]\tilde{E},$$

which admits normal mode solutions of the form $\tilde{E} = Ee^{\omega t}$, where the eigenvalue ω is given by

$$\omega = -\frac{1}{\tau_E}\left(\frac{a_p}{D_{n0}n}\right)\sum_a Z_a \frac{\partial \Gamma_a}{\partial \bar{E}}(E_0). \tag{4}$$

If $\omega < 0$, then the steady-state solution is stable; if $\omega > 0$, then it is unstable. We note that τ_p is a measure of the particle confinement time, which is roughly the time scale over which n_a and T_a change. For typical parameters ($B_0 = 10^4$G, $a_p = 20$ cm, $T = 300$ eV), we have $\tau_E \ll \tau_p$; hence, from Eq. (4) we deduce that for $(a_p/D_{n0}n)\sum_a Z_a \partial\Gamma_a/\partial\bar{E} = O(1)$ the rate of relaxation of the ambipolarity equation to a stable steady-state is much faster than the rate at which the density and temperature are changing. Therefore, we consider only the steady-state version of Eq. (1). Physically we have used the fact that the time for adjustment of the electric field is much faster than the time for changes in the density and temperature. This enables us to reduce the electric field equation from a first-order differential equation coupled to the density and temperature equation to an algebraic equation on each flux surface solved with the instantaneous values of n_a and T_a.

In Fig. 1, we plot the electron and ion fluxes against Φ' for three typical situations with n and T_e, T_i fixed. This is meaningful as long as the eletric field equation, Eq. (1), relaxes much faster than the density and temperature equations. In Fig. 1(a) we show what the fluxes are like when there is only one electric field solution which is negative. This is called the ion root and corresponds to the magnetically confined electrons electrostatically retarding the ions. In Fig. 1(b) we show three possible solutions. The most positive electric field one is called the electron root and corresponds to magnetically confined ions electrostatically retarding the electrons. The root between the electron and ion roots is called the unstable root, since it is clear that a small perturbation away from it will grow. In Fig. 1(c) we show a situation where only the electron root exists. The one situation where it is not meaningful to consider this equation for fixed n and T occurs when the unstable root is close to either of the stable roots (the ion or the electron root).

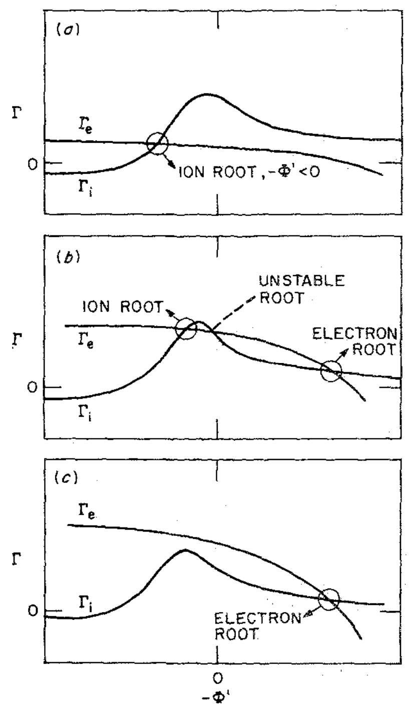

FIG. 1. Ion flux Γ_i and electron flux Γ_e vs electric field $-\Phi'$: (a) ion root only; (b) electron, ion, and unstable roots; (c) electron root only.

Then the relaxation time to the stable solution close to the unstable root becomes very long ($\omega \to 0$) and the electric field equation can couple effectively to the density and temperature equations, which change on the diffusion (τ_p) time scale.

The electric field equation [Eq. (1)] is a nonlinear first-order equation in time and, for an arbitrary initial condition, will relax away from the unstable solution and towards a stable solution. Hence, in Fig. 1(b), if the initial condition falls to the right of the unstable root, then the steady-state solution will be the electron root. Conversely, if the initial condition falls to the left of the unstable root, then the steady-state solution will be the ion root. This equation will always choose one if the stable solutions but may give rise to discontinuous electric field profiles. This can happen under two circumstances. The first is that at some radial location only an ion (electron) root exists and at another close point only an electron (ion) root exists. The second is that the initial conditions are such that, where there are two stable roots, then at a given point one of the roots is the solution and at a neighboring point the the other is. The second situation, although allowed by the equation, is unphysical, since what was neglected in obtaining the fluxes was the coupling between flux surfaces due to the finite orbit width. This leads to the differential term, which has been derived in [3]. This term will couple the two flux surfaces together in the second situation and make it very likely that if one is on the ion (electron) root then the other will be on the same root. In the first situation there is no choice, and what the differential coupling will do is provide a finite width to the root jumping region. We can express the preceding ideas mathematically by noting that from the steady-state solution of Eq. (1) we can write $\Phi' = \Phi'(n_a, T_a, n'_a, T'_a)$. If at some point defined by (n_a, T_a, n'_a, T'_a) the solution is Φ', then the solution at a neighboring point $(n_a + \Delta n_a,\ T_a + \Delta T_a,\ n'_a + \Delta n'_a,\ T'_a + \Delta T'_a)$ will be given by

$$\Phi'(n_a + \Delta n_a, T_a + \Delta T_a, n'_a + \Delta n'_a, T'_a + \Delta T'_a)$$
$$\simeq \Phi'(n_a, T_a, n'_a, T'_a) + \Delta n_a \frac{\partial \Phi'}{\partial n_a} + \Delta n_a \frac{\partial \Phi'}{\partial n_a} + \Delta T_a \frac{\partial \Phi}{\partial T_a} + \Delta n'_a \frac{\partial \Phi'}{\partial n'_a} + \Delta T'_a \frac{\partial \Phi}{\partial T'_a}. \tag{5}$$

In Eq. (5), we have Taylor-expanded with the implicit assumption that all the partial derivatives of Φ' are bounded in the region (n_a, T_a, n'_a, T'_a) to $(n_a + \Delta n_a, T_a + \Delta T_a, n'_a + \Delta n'_a, T'_a + \Delta T'_a)$. If this is so, then the electric field at $(n_a + \Delta n_a, ...)$ should be close to the electric field at $(n_a, ...)$. This assumption breaks down if $\Phi'(n_a + \Delta n_a, ...)$ or $\Phi'(n_a, ...)$ is very close to or on the unstable solution to the ambipolarity relationship. In this case, at $(n_a, ...)$ only one root exists, while at $(n_a + \Delta n_a, ...)$ only the other root exists.

The density and temperature equations are diffusion equations that are solved using an implicit Crank–Nicholson scheme. The density equation is

$$\frac{\partial n_a}{\partial t} + \frac{1}{V'(r)} \frac{\partial}{\partial r} [V'(r)\, \Gamma_a] = S_a$$

where $V(r)$ is the volume inside a pressure surface labelled by r and S_a is the source of particles of species "a." The electron temperature is given by

$$\frac{\partial}{\partial t}\left(\frac{3}{2} n_e T_e\right) + \frac{1}{V'(r)} \frac{\partial}{\partial r}\left[\left(V'(r)\left(q_e + \frac{5}{2} T_e \Gamma_e\right)\right]\right] = \Phi' \Gamma_e + S_{E_e}$$

where q_e is the random heat flux and S_{E_e} contains all the sources and sinks of energy to the electrons. These include ECH heating, electron–ion energy exchange, and losses due to electron impact ionization of neutrals. The ion temperature is given by

$$\frac{\partial}{\partial t}\left(\tfrac{3}{2} \sum_j n_j T_i\right) + \frac{1}{V'(r)} \frac{\partial}{\partial r}\left[V'(r)\left(q_i + \tfrac{5}{2} T_i \sum_j \Gamma_i\right)\right] = -\Phi' \sum_j Z_j \Gamma_j + S_{E_i}$$

where the sum $\sum_j$ is over all ion species that have an assumed common temperature T_i. The term S_{E_i} contains all the sources and sinks of energy to the ions. These include charge exchange losses, ICH heating, and ion–electron energy exchange. We obtain the electric field at the point r and at time t by solving the steady-state ambipolarity equation

$$\sum_a Z_a \Gamma_a[\Phi'(r, t), n_a(r, t), T_a(r, t)] = 0, \tag{6}$$

using a bisection root finder ZEROIN [5]. The choice of this root finder ensures convergence. The bisection interval containing the root is obtained in the following manner.

We solve Eq. (6) from the center of the plasma to the edge. If Δr is the finite difference grid spacing and $[a, b]$ is the bisection interval containing the root of Eq. (6) at (r, t), then we set

$$a = \Phi'(r - \Delta r, t), \tag{7}$$

$$b = \Phi'(r - \Delta r, t) - j s_t s_g. \tag{8}$$

At $r = 0$ (the center of the plasma) we set $a = 0$ since we expect the electric field to vanish at the center of the plasma from symmetry considerations (flux cannot accumulate at the plasma center). In Eq. (8), j is the smallest integer such that

$$\left[\sum_a Z_a \Gamma_a(\Phi' = a)\right]\left[\sum_a Z_a \Gamma_a(\Phi' = b)\right] < 0;$$

s_t is a step size, and s_g is given by

$$s_g = \operatorname{sign}\left\{\sum_a Z_a \Gamma_a[\Phi'(r - \Delta r, t)]\right\}.$$

This choice of bisection interval will always bracket a stable root, as can be seen from Fig. 1(b). If $\Phi'(r-\Delta r, t)$ falls to the left of the ion root, then $s_g < 0$ and point b will fall to the right of the ion root. If $\Phi'(r-\Delta r, t)$ falls to the right of the ion root but to the left of the unstable root, then $s_g > 0$ and point b will fall to the left of the ion root. Similarly, if $\Phi'(r-\Delta r, t)$ falls to the right of the unstable root but to the left of the electron root, then $s_g < 0$ and point b will fall to the right of the electron root. Finally, if $\Phi'(r-\Delta r, t)$ falls to the right of the electron root, then $s_g > 0$ and point b will fall to the left of the electron root. Since both ends of the besection interval are related to $\Phi'(r-\Delta r, t)$, if the unstable root is more than s_g away from the stable roots, then the root found will be the one closest to $\Phi'(r-\Delta r, t)$. This tends to build in the physical notion that flux surfaces are coupled together through the finite orbit width. If the unstable root is closer than s_g to one of the stable roots, then this way of choosing the bisection interval will bracket the other stable root. Hence, in Fig. 1(b), if the unstable root and the ion root are about to coalesce, then the finite step size s_g will cause the electron root to be bracketed. In addition, in this case it may be that $\Phi'(r-\Delta r, t)$ falls between the unstable root and the electron root. Then the electron root will be bracketed. This numerical root jumping will only occur when the unstable root is very close to a stable root and can be regarded as simulating the effect of noise on the choice of the root. It has been shown [6] that when the unstable root is close to a stable root, then the solution is very sensitive to small fluctuations (which occur in all real plasmas) and has a high probability of evolving to the other stable root.

This choice of the bisection interval in combination with a bisection root finder will always find a stable solution and will tend to find a spatially continuous solution. We remark that this procedure at the finite-difference level is conceptually the same as solving a first-order differential equation for the electric field implicitly, where the derivative of the electric field is multiplied by a small parameter. This small parameter can be regarded as the finite orbit deviation from the flux surface.

We comment that this procedure does not exclude the possibility of a shock-like structure for the electric field. This will happen when only the ion root exists at one radial location and only the electron root exists at a neighboring location. This procedure will exclude such a possibility when a continuous solution can also be found.

III. Application to a Bumpy Torus Plasma

To illustrate the electric field algorithm, the equations were solved for a bumpy torus plasma. The configuration, diffusion coefficients, and equations used were the same as in [7], except that the electric field was obtained self-consistently. The density and temperature equations were evolved to steady-state conditions for a fixed volume-averaged density, volume-averaged electron temperature, and volume-averaged ion temperature.

In Fig. 2, we plot the potential profile against major radius and volume-averaged ion temperature. The volume-averaged density and electron temperature were kept fixed in these figures. These plots were motivated by experimental observations that the polarity of the potential changes as a function of ion temperature [8]. In Fig. 2(a) we have $\langle n \rangle = 5 \times 10^{12}$ cm^{-3}, $\langle T_e \rangle = 140$ sV ($\langle ... \rangle = (1/V) \int dV ...$), and we see that the potential for small ion temperatures is a hill; then, as the ion temperature increases, it changes abruptly to a well at every spatial point. For low ion temperatures the electric field is positive because for these parameters the ions are collisional and the electrons are collisionless. Hence the electron loss rate exceeds the ion loss rate, thus forcing the electric field to be positive. As the ion temperature is increased, the ions become collisionless and the ion loss rate begins to exceed the electron loss rate. This forces the electric field to be negative everywhere to retard the lossy ions. In Fig. 2(b) we take $\langle n \rangle = 5 \times 10^{11}$ cm^{-3}, $\langle T_e \rangle = 50$ eV, and change the ion temperature from 15 to 25 eV. As before, the potential is a hill, but as the ion temperature increases it becomes a well, first in the center, that then spreads toward the edge as the temperature increases. This is because at the edge of the plasma the temperature is low and the edge ions stay collisional after the center has jumped roots.

In Fig. 3, we show a typical case where the electric field was positive everywhere (a potential hill), but a negative electric field solution also existed at the edge of the plasma. The negative electric field solution was found by taking one end of the bisection interval to be large negative value. This shows another advantage of using this root finder approach: it is easy to check for other possible solutions by varying the initial guess. In this case, the electric field was taken to be the one that was positive everywhere, which is also the one that has no discontinuity in the solution.

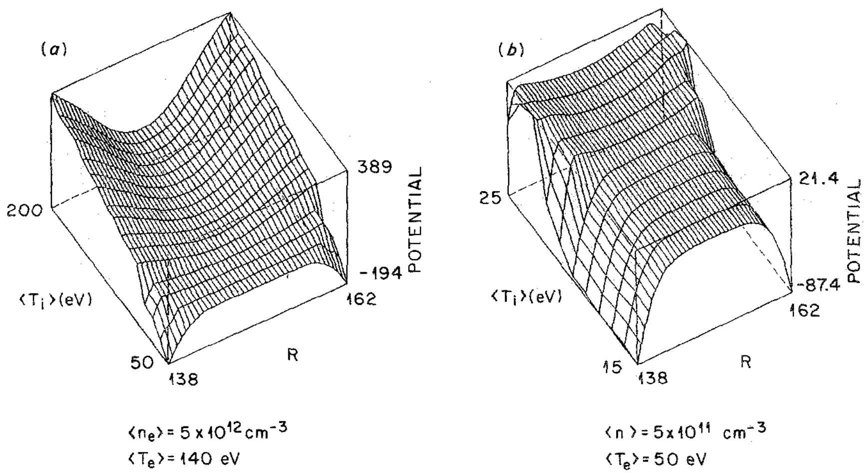

FIG. 2. Potential Φ vs $\langle T_i \rangle$ and major radius R: (a) $\langle n \rangle = 5 \times 10^{12}$ cm^{-3}, $\langle T_e \rangle = 140$ eV; (b) $\langle n \rangle = 5 \times 10^{11}$ cm^{-3}, $\langle T_e \rangle = 50$ eV.

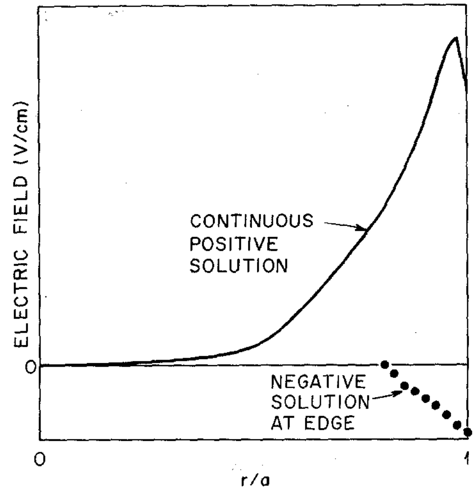

FIG. 3. Electric field $-\Phi'$ vs normalized radius for a continuous positive solution with a negative solution at the edge. The plasma parameters are $\langle T_e \rangle = 50$ eV, $\langle n \rangle = 5 \times 10^{11}$ cm^{-3}, $\langle T_i \rangle = 18$ eV.

In Fig. 4, we show a case for which a positive solution was found at the edge of the plasma, in addition to the continuous negative solution. As in Fig. 3, the algorithm chose the continuous solution.

These results on the change of the polarity of the potential are summrized in Fig. 5. We plot the boundary across which the potential changes in the $\langle T_e \rangle$, $\langle T_i \rangle$ plane for fixed $\langle n \rangle$. In Fig. 5, we have $\langle n \rangle = 5 \times 10^{11}$, 5×10^{12}, and 5×10^{13} cm^{-3}. On the right side of each curve the potential is positive, while on the left side it is

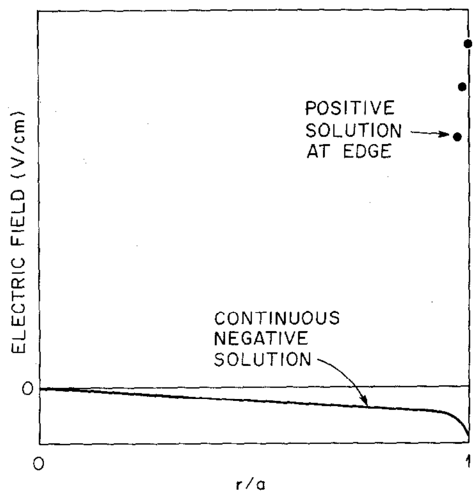

FIG. 4. Electric field $-\Phi'$ vs normalized radius for a continuous negative solution with a positive solution at the edge. The plasma parameters are $\langle T_e \rangle = 140$ eV, $\langle n \rangle = 5 \times 10^{12}$ cm^{-3}, $\langle T_i \rangle = 100$ eV.

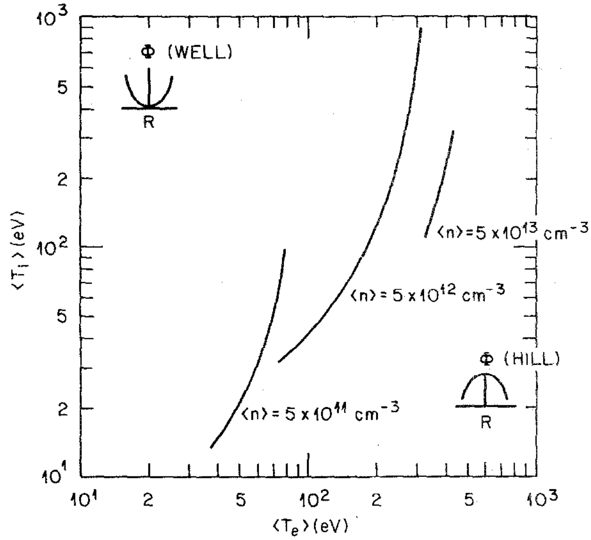

FIG. 5. Boundaries in $\langle T_i \rangle$, $\langle T_e \rangle$ plane across which potential polarity changes.

negative. As the density increases, the curve moves to the right because the collisionality increases. This means that at a given ion temperature the electron temperature has to increase in order for the electron collisionality to drop into the collisionless regime and give lossy electrons. These curves become almost parallel to the $\langle T_i \rangle$ axis for large $\langle T_i \rangle$ and indicate that a jump to the electron root for large $\langle T_i \rangle$ is largely facilitated by increasing $\langle T_e \rangle$, not $\langle T_i \rangle$. This would indicate that electron heating is desirable for encouraging the formation of the electron root, which has been predicted to be the more favorable root [2].

IV. CONCLUSION

We have described a method for obtaining the electric field in a nonaxisymmetric device that will tend to give spatially continuous, temporally stable electric fields. We have applied it to a bumpy torus, and we have obtained boundaries in the $\langle T_e \rangle$, $\langle T_i \rangle$ space across which the potential changes polarity for a given density. These plots are useful for indicating what sort of heating could be useful in encouraging the transition of the plasma to one or the other stable root.

Future areas of research to consider are the effect of direct losses on the radial electric field and the effect of rf-driven diffusion. Direct losses may introduce integral expressions for the electric field; rf diffusion will depend also on the microwave power and how that changes the lossiness of each species.

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